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A LOCAL BOUNDING PRINCIPLE FOR DISSIPATION ENERGY IN SHAKEDOWN OF ELASTIC-PERFECTLY PLASTIC SOLIDS†

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Abstract—The paper deals with elastic–perfectly plastic solids which shake down to a purely elastic state under the action of quasi-static loads which vary arbitrarily remaining within a given domain. It presents a technique for bounding the dissipation energy which is done locally at a given point of the body during the adaptation process. This technique leads to a problem of finite plasticity which may be considered as the holonomic version of the original shakedown problem suitably perturbed. The holonomic solution can be directly used to construct the local bound which proves to be also the most stringent one. A simple example illustrates the method.

1. INTRODUCTION

The bounding techniques play a crucial role in the assessment of structural safety within the framework of shakedown theory. Among the post-shakedown deformation parameters to be bounded, plastic dissipation energy is an important one. The bounding theorems for dissipation energy formulated so far (see, e.g. [1–5]) consider the dissipation energy of the overall structure, while the interest is in a bounding theorem for the energy dissipated locally at any point of the structure. In certain narrower circumstances, the known bounds on the so-called "plastic strain intensities" (or on functions of them) can be considered as local bounds on dissipation energy associated with specific deformation modes (see, e.g. [6–8]). However, to the author's knowledge, a true local bound on plastic dissipation energy has not been formulated so far.

The present paper, following a perturbation method adopted by the same author elsewhere[9–11], furnishes a bounding technique for the local plastic dissipation which is produced during the application of external actions (repeated loads and temperature cycles). Dynamic effects as well as workhardening are considered as negligible and displacements are treated as infinitesimal.

By the use of an argument which is similar but not identical to that used by Koiter in his famous article on plasticity[1], we obtain a bound expression which, being dependent on some free parameters, may be optimized. This leads us to formulate a constrained minimization problem, whose optimality conditions describe a boundary value problem of finite plasticity, which appears as the holonomic version of the original shakedown problem suitably perturbed. The holonomic solution permits the bound to be the most stringent and expressed in local terms, in the sense that the bounding quantity is referred to the same body portion which the dissipation energy is referred to.

The usual conventions of tensor calculus are adopted, such as the summation convention for repeated indices. Commas indicate derivatives with respect to coordinates, i.e. $(\ldots)_{,i} = \partial(\ldots)/\partial x_i$, while the time derivative is indicated by a superimposed dot.

2. THE SHAKEDOWN PROBLEM

An elastic–perfectly plastic solid of volume V is referred to a rectangular cartesian coordinate system, say $x = (x_1, x_2\ x_3)$. Its surface S is restrained on the part S_2, where the displacements $u_i = U_i$ are prescribed; while on the complementary part $S_1 = S - S_2$ the tractions T_i, $(i = 1, 2, 3)$, are applied. Body forces F_i, as well as imposed strains ϑ_{ij} (for instance of thermal origin) are supposed to be present.

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The above external actions, which vary with time t through a time-sequence which is unknown, belong to a given *loading domain* Π, or, in other words, they can be considered as one-to-one functions of the point τ_ν, $(\nu = 1, 2, \ldots, r)$, of an r-dimensional domain, as in the following

$$\left.\begin{array}{ll} T_i = T_i(\tau_\nu), & \text{on } S_1 \\ F_i = F_i(\tau_\nu), & \text{in } V \\ U_i = U_i(\tau_\nu), & \text{on } S_2 \\ \vartheta_{ij} = \vartheta_{ij}(\tau_\nu), & \text{in } V \end{array}\right\rangle \tau_\nu \in \Pi, \ (\nu = 1, 2, \ldots, r). \qquad (2.1\text{a–d})$$

A time function $\tau_\nu = \tau_\nu(t)$, $t \geq 0$, determines a possible loading history provided $\tau_\nu(t) \in \Pi$ for every $t \geq 0$.

Denoting the stress tensor by σ_{ij}, the equilibrium equations are

$$\sigma_{ij,j} + F_i = 0, \quad \text{in } V, \qquad (2.2\text{a})$$

$$\sigma_{ij} n_j = T_i, \quad \text{on } S_1, \qquad (2.2\text{b})$$

where n_j is the unit external normal of S, while the compatibility equations are

$$\epsilon_{ij} = e_{ij} + p_{ij} + \vartheta_{ij}, \quad \text{in } V, \qquad (2.3\text{a})$$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{in } V, \qquad (2.3\text{b})$$

$$u_i = U_i, \quad \text{on } S_2, \qquad (2.3\text{c})$$

where ϵ_{ij} is the (total) strain tensor, e_{ij} its elastic part and p_{ij} its plastic part.

The elastic behaviour of the material is described by Hooke's law, i.e.

$$e_{ij} = A_{ijhk}\sigma_{hk}, \quad \text{in } V, \qquad (2.4)$$

where A_{ijhk} is the usual elastic coefficient tensor, as well as by the elastic domain

$$f_\alpha(\sigma_{ij}) \leq 0, \quad \forall \alpha \in (1, 2, \ldots, m), \quad \text{in } V, \qquad (2.5)$$

where the convex yield functions

$$f_\alpha = f_\alpha(\sigma_{ij}), (\alpha = 1, 2, \ldots, m), \quad \text{in } V, \qquad (2.6)$$

are independent of plastic strain and play the role of plastic potentials. Therefore, the plastic strain rate tensor is given by the usual flow-rule

$$\dot{p}_{ij} = \frac{\partial f_\alpha}{\partial \sigma_{ij}} \dot{\lambda}_\alpha, \quad \text{in } V, \qquad (2.7)$$

along with the following side equations

$$f_\alpha \leq 0, \ \dot{\lambda}_\alpha \geq 0, \quad \forall \alpha \in (1, 2, \ldots, m), \quad \text{in } V, \qquad (2.8\text{a, b})$$

$$f_\alpha \dot{\lambda}_\alpha = 0, \quad \dot{f}_\alpha \dot{\lambda}_\alpha = 0, \quad \text{in } V. \qquad (2.9\text{a, b})$$

The plastic strain intensities $\lambda_\alpha = \lambda_\alpha(x, t)$, given by

$$\lambda_\alpha = \int_0^t \dot{\lambda}_\alpha \, \mathrm{d}t, \quad (\alpha = 1, 2, \ldots, m), \qquad (2.10)$$

as well as the plastic work density $W = W(x, t)$, given by

$$W = \int_0^t \sigma_{ij}\dot{p}_{ij}\,\mathrm{d}t, \tag{2.11}$$

are nonnegative functions of t for any $x \in V$. If they remain finite everywhere in V while t increases, the structure shakes down to a purely elastic state and, as we say, it adapts to the loads. Shakedown (or adaptation) criteria enable us to recognize if the structure is able to shake down (or to adapt). However we know that such information is insufficient: the structure, in fact, may fail because the parameters (2.10) and (2.11), though finite, are excessive with respect to given safety criteria. The loading history being unknown and a direct evaluation of the above parameters being thus impossible, we are faced with the problem of finding *a priori* bound on their post-shakedown values. Bounds on plastic strain intensities λ_α have already been given[6–11], while bounds on plastic work density W have not been formulated so far. We will give a bound on W in the following paragraphs.

3. AN ASSOCIATED BOUNDARY VALUE PROBLEM

Let us consider a fictitious elastic–perfectly plastic solid which is identical to that of Section 2, except that its yield functions, instead of (2.6), are

$$\bar{f}_\alpha(\sigma_{ij}) \equiv f_\alpha(g\sigma_{ij}), \quad (\alpha = 1, 2, \ldots, m), \quad \text{in } V, \tag{3.1}$$

where the coefficient g is an arbitrary scalar function (perturbation function) such that

$$g = g(x) \geq 1, \forall x \in V. \tag{3.2}$$

Moreover, this solid is in an arbitrary state of "initial" plastic strains p^*_{ij} with associated self-stresses σ^*_{ij}, both independent of t.

The elastic response to the given external actions satisfies the following equations

$$\left.\begin{array}{ll} \bar{\sigma}_{ij,j} + F_i = 0, & \text{in } V, \\ \bar{\sigma}_{ij}n_j = T_i, & \text{on } S_1, \end{array}\right\rangle \text{(equilibrium)} \tag{3.3a, b}$$

$$\left.\begin{array}{ll} \bar{\epsilon}_{ij} = \bar{e}_{ij} + \vartheta_{ij} + p^*_{ij}, & \text{in } V, \\ \bar{\epsilon}_{ij} = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i}), & \text{in } V, \\ \bar{u}_i = U_i, & \text{on } S_2, \end{array}\right\rangle \text{(compatibility)} \tag{3.4a–c}$$

$$\bar{e}_{ij} = A_{ijhk}\bar{\sigma}_{hk}, \quad \text{in } V. \quad \text{(Hooke's law)} \tag{3.5}$$

The solution to this set of equations is statically admissible if it does not violate the yield condition, i.e.

$$\bar{f}_\alpha(\bar{\sigma}_{ij}) \leq 0, \quad \forall \alpha \in (1, 2, \ldots, m), \quad \text{in } V, \tag{3.6}$$

or, in view of eqn (3.1),

$$f_\alpha(g\bar{\sigma}_{ij}) \leq 0, \quad \forall \alpha \in (1, 2, \ldots, m), \quad \text{in } V, \tag{3.7}$$

where the stresses $\bar{\sigma}_{ij}$ can be decomposed into the elastic stresses σ^E_{ij} (elastic solution to the original problem) and the self-stresses σ^*_{ij}. We have in fact

$$\bar{\sigma}_{ij} = \sigma^E_{ij} + \sigma^*_{ij}, \tag{3.8a}$$

$$\bar{\epsilon}_{ij} = \epsilon^E_{ij} + \epsilon^*_{ij}, \tag{3.8b}$$

$$\bar{u}_i = u^E_i + u^*_i, \tag{3.8c}$$

with obvious meanings of the symbols.

Suppose a statically admissible solution exists. Then, in virtue of Drucker's postulate[1], we have

$$(g\bar{\sigma}_{ij} - \sigma_{ij})\dot{p}_{ij} \le 0, \qquad \text{in } V, \tag{3.9}$$

or, equivalently,

$$(g-1)\sigma_{ij}\dot{p}_{ij} \le g(\sigma_{ij} - \bar{\sigma}_{ij})\dot{p}_{ij}, \qquad \text{in } V, \tag{3.10}$$

which can be written in the form

$$(g-1)\sigma_{ij}\dot{p}_{ij} \le G(\sigma_{ij} - \bar{\sigma}_{ij})\dot{p}_{ij}, \qquad \text{in } V, \tag{3.11}$$

provided that G is a constant satisfying the inequality

$$G \ge g(x), \quad \forall\, x \in V. \tag{3.12}$$

From eqns (2.3a) and (3.4a) and from Hooke's law we deduce the equality

$$\dot{p}_{ij} = \dot{\epsilon}_{ij} - \dot{\bar{\epsilon}}_{ij} - A_{ijhk}(\dot{\sigma}_{hk} - \dot{\bar{\sigma}}_{hk}), \tag{3.13}$$

in view of which inequality (3.11) becomes, after an integration over the volume V,

$$\int_V (g-1)\sigma_{ij}\dot{p}_{ij}\,\mathrm{d}V \le G\left[\int_V (\sigma_{ij} - \bar{\sigma}_{ij})(\dot{\epsilon}_{ij} - \dot{\bar{\epsilon}}_{ij})\,\mathrm{d}V - \int_V A_{ijhk}(\sigma_{ij} - \bar{\sigma}_{ij})(\dot{\sigma}_{hk} - \dot{\bar{\sigma}}_{hk})\,\mathrm{d}V\right]. \tag{3.14}$$

Since the first integral in the right-hand member is zero, we finally have

$$\int_V (g-1)\sigma_{ij}\dot{p}_{ij}\,\mathrm{d}V \le -G\dot{B}(t), \tag{3.15}$$

where $B(t)$ is the nonnegative functional

$$B(t) = \frac{1}{2}\int_V A_{ijhk}(\sigma_{ij} - \bar{\sigma}_{ij})(\sigma_{hk} - \bar{\sigma}_{hk})\,\mathrm{d}V. \tag{3.16}$$

An integration over the time interval $(0, t_1)$, remembering the definition (2.11) and cancelling the subtractive nonnegative term $B(t_1)$, furnishes

$$\int_V (g-1)W(x, t_1)\,\mathrm{d}V \le GB(0) \tag{3.17}$$

where, in consideration that $\sigma_{ij}(x,0) = \sigma^E_{ij}(x,0) = 0$ everywhere in V, $B(0)$ proves to be the functional

$$B(0) = \frac{1}{2}\int_V A_{ijhk}\sigma^*_{ij}\sigma^*_{hk}\,\mathrm{d}V. \tag{3.18}$$

Inequality (3.17) is a bound on plastic dissipation energy. This bound, if we take $g(x) =$ const. $= G$, transforms into the bound on the overall plastic work given in[1].

4. A MINIMUM PRINCIPLE

In order to make the bound (3.17) the most stringent, we may search for the best choice of the self-stresses σ^*_{ij} subject to the condition that the elastic solution $\bar{\sigma}_{ij} = \sigma^E_{ij} + \sigma^*_{ij}$ satisfies eqns (3.7). For the sake of a greater generality, we set

$$g = \gamma + 1, \qquad \text{in } V, \tag{4.1}$$

and let the net perturbation function $\gamma = \gamma(x)$ be defined within a positive multiplier ω, i.e.

$$\gamma = \omega\bar{\gamma}, \qquad \bar{\gamma} = \bar{\gamma}(x) \geq 0, \qquad \omega > 0. \tag{4.2a–c}$$

Here $\bar{\gamma}(x)$ is supposed to be arbitrarily prescribed in V, while ω is unknown. With these transformations, inequality (3.17) can thus be written

$$\int_V \bar{\gamma} W(x, t_1)\,\mathrm{d}V \leq \frac{G}{\omega} B(0). \tag{4.3}$$

Let us now consider the following minimization problem:

$$\min \Phi = \frac{G}{\omega}\int_V \frac{1}{2} A_{ijhk}\rho_{ij}\rho_{hk}\,\mathrm{d}V, \tag{4.4a}$$

subject to $\omega > 0$ and

$$\rho_{ij,j} = 0, \qquad \text{in } V, \tag{4.4b}$$

$$\rho_{ij}n_j = 0, \qquad \text{on } S_1, \tag{4.4c}$$

$$\varphi_\alpha(s_{ij}) \leq 0, \qquad \text{in } V + \Pi, \tag{4.4d}$$

$$g - G \leq 0, \qquad \text{in } V, \tag{4.4e}$$

where the following substitutions are to be operated:

$$g = \omega\bar{\gamma}(x) + 1, \qquad \varphi_\alpha = f_\alpha(gs_{ij}), \qquad \text{in } V, \tag{4.5a, b}$$

$$s_{ij} = \sigma^E_{ij}(x, \tau_\nu) + \rho_{ij}(x), \qquad \text{in } V + \Pi. \tag{4.5c}$$

The problem (4.4a–e) is a convex problem of calculus of variations with equality and inequality constraints. It can be treated following a classical path (see Appendix). The optimality conditions prove to be the following

$$\rho_{ij,j} = 0, \qquad \text{in } V, \tag{4.6a}$$

$$\rho_{ij}n_j = 0, \qquad \text{on } S_1, \tag{4.6b}$$

$$\frac{1}{2}(v_{i,j} + v_{j,i}) = A_{ijhk}\rho_{hk} + q_{ij}, \qquad \text{in } V, \tag{4.7a}$$

$$v_i = 0, \qquad \text{on } S_2, \tag{4.7b}$$

$$\left.\begin{array}{llll} \varphi_\alpha = f_\alpha(gs_{ij}), & s_{ij} = \sigma^E_{ij} + \rho_{ij}, & & \\ \varphi_\alpha \leq 0, & l_\alpha \geq 0, & \varphi_\alpha l_\alpha = 0, & d_{ij} = \dfrac{\partial\varphi_\alpha}{\partial s_{ij}} l_\alpha, \end{array}\right\rangle \text{in } V + \Pi \tag{4.8a–d}$$

(4.7a, b)

$$q_{ij} = \int_\Pi d_{ij}\,\mathrm{d}\Pi, \qquad D = \int_\Pi s_{ij}d_{ij}\,\mathrm{d}\Pi, \qquad \text{in } V \tag{4.9a–b}$$

$$\int_V \frac{1}{2} A_{ijhk}\rho_{ij}\rho_{hk}\,\mathrm{d}V = \omega G \int_V \bar{\gamma} g^{-1} D\,\mathrm{d}V, \tag{4.10}$$

$$G = \max_{x \in V} g(x), \tag{4.11}$$

where g is given by eqns (4.1) and (4.2a).

The set of eqns (4.6a, b)–(4.11) may be interpreted as the holonomic description of the original shakedown problem suitably perturbed. This perturbation consists in a homothetic shrinkage in the ratio $1/g$ of the yield surfaces. Since the stresses ρ_{ij} are the elastic response to the imposed strains q_{ij}, we find that the best choice for σ^*_{ij} and thus for p^*_{ij} is

$$\sigma^*_{ij} = \rho_{ij}, \quad p^*_{ij} = q_{ij}, \qquad \text{in } V. \tag{4.12a, b}$$

The bound optimization can be performed in two different but equivalent ways: one is solving the holonomic boundary value problem (4.6a, b)–(4.11), the other is solving the minimization problem (4.4a–e) with the side equations (4.5a–c). The latter constitutes in fact a minimum principle which characterizes the holonomic solution and hence the optimal bound.

In view of the optimality condition (4.10) and the eqn (4.12a), the bound (4.3) can be given a different form, i.e.

$$\int_V \bar{\gamma} W(x, t_1)\,\mathrm{d}V \le G^2 \int_V \bar{\gamma} g^{-1} D(x)\,\mathrm{d}V. \tag{4.13}$$

Let us now make the following choice for $\bar{\gamma}$:

$\bar{\gamma} = 1$, within a given volume element ΔV,
$\bar{\gamma} = 0$, in $V - \Delta V$,

so that $g = 1 + \omega$ in ΔV and $g = 1$ in $V - \Delta V$. Since $G = 1 + \omega$ in accordance with eqn (4.11), inequality (4.13) takes the significant format

$$W(\hat{x}, t_1) \le (1 + \omega) D(\hat{x}), \qquad \text{in } V. \tag{4.14}$$

Equation (4.14) is valid for any subsequent time $t_1 > 0$ and for any point $\hat{x} \in V$ which the plastic work density is referred to. The quantity D represents the local dissipation energy associated with the holonomic solution mentioned earlier, which is to be computed in the same reference point $\hat{x}$.

Inequality (4.14) is the desired bound on plastic work density W, at any point of the body. For every such point this bound requires the solution of an analysis problem of finite plasticity, while a bound on the overall plastic work like that given in [1] requires only an elastic solution. The more information one can get through the application of inequality (4.14) is thus to be paid for with a greater computational effort.

5. EXAMPLE

As just an example, we consider the clamped beam of length $2L$ (Fig. 1a), whose elastic–perfectly plastic behaviour is characterized by the yield moments

$$M_y^- = M_y, \quad M_y^+ = \beta M_y, \quad (\beta > 1). \tag{5.1a, b}$$

A point load is applied on the middle section and its intensity F varies quasi-statically as

$$F = F_0 \tau, \quad \tau \in \Pi, \quad \Pi = \{\tau:\ 0 \le \tau \le s\} \tag{5.2}$$

where F_0 is the elastic limit load, i.e.

$$F_0 = 4M_y/L. \tag{5.3}$$

In view of the symmetry, the self-stresses which arise in the beam can be described in terms of two equal and opposite couples, say X, applied at the ends of the simply supported beam (Fig. 1b). For $s \le 1$ the system is in the elastic domain, while for $1 < s < s_c$ it is out of the elastic domain but able to adapt to the loading at the price of plastic deformations. We want to find a bound on the plastic work which is dissipated at one of the clamped end sections.

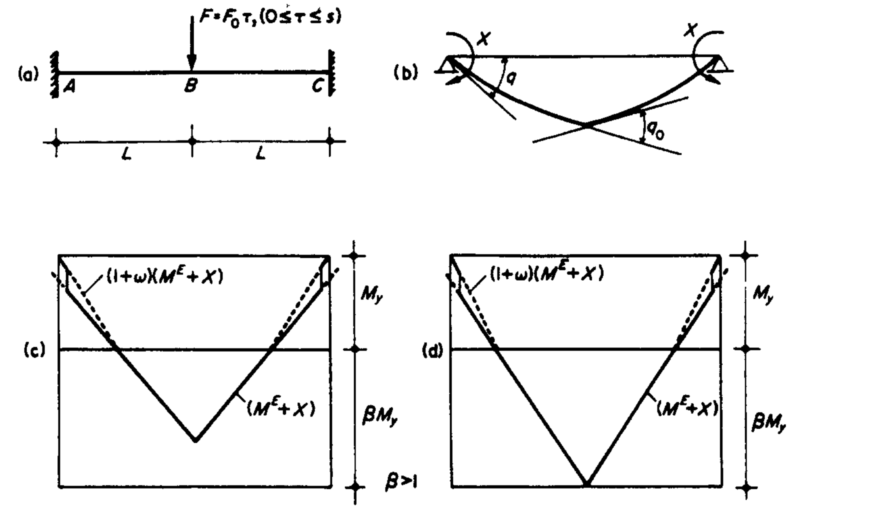

Fig. 1. Clamped beam loaded on the middle section: (a) geometrical and loading sketch; (b) deformation pattern; (c) perturbed bending moments for $1 \le s \le (\beta+2)/3$; (d) perturbed bending moments for $(\beta+2)/3 \le s \le (\beta+1)/2$.

5.1 *Real plastic work*

The real plastic work W, which is done locally at the extreme point A, is expressed by the product $M_y|p|$, p being the plastic rotation at A. Since $p = -F_0L^2(s-1)/4EI$, it follows:

$$4W/F_0L = a(s-1), \quad (1 \le s \le s_c) \tag{5.4}$$

where we have set

$$a = F_0L^2/4EI, \tag{5.5}$$

and eqn (5.3) has been accounted for. The value s_c is the plastic collapse load which proves to be

$$s_c = (\beta+1)/2. \tag{5.6}$$

5. BOUND ON PLASTIC WORK

For the evaluation of the bound we will apply the optimality conditions of Section 4. We consider the total bending moment $M^E + X$, which is amplified in the ratio $1/(1+\omega)$ in the vicinities of the end sections A and C. The yield conditions are

$$\textit{at Section A:} \quad -\left(X - \frac{1}{4}F_0Ls\right)(1+\omega) - M_y \le 0, \tag{5.7a}$$

$$\textit{at Section B:} \quad \left(X + \frac{1}{4}F_0Ls\right) - \beta M_y \le 0, \tag{5.7b}$$

which can also be written in the following dimensionless form

$$\textit{at Section A:} \quad (s-\xi)(1+\omega) - 1 \le 0, \tag{5.8a}$$

$$\textit{at Section B:} \quad (s+\xi) - \beta \le 0, \tag{5.8b}$$

where we have introduced the self-stress parameter

$$\xi = 4X/F_0L. \tag{5.9}$$

The complementary strain energy B_0 associated with the self-stresses is

$$B_0 = LX^2/EI = (F_0L/4)a\xi^2, \tag{5.10}$$

while the plastic rotation q at the end section (Fig. 1b) is

$$\textit{no yielding at } B\text{:}\quad q = -XL/EI = -a\xi, \tag{5.11a}$$

$$\textit{yielding at } B\text{:}\quad q = -\left(a\xi + \frac{1}{2}q_0\right), \tag{5.11b}$$

q_0 being the relative rotation at B. In accordance with eqn (4.10), $\bar{\gamma}$ being equal to one in the vicinities of the end sections and equal to zero in all the remaining part of the beam, we write the equality

$$a\xi^2 = 2\omega(\xi - s)q, \tag{5.12}$$

where we have set

$$4D/F_0L = (\xi - s)q, \quad G = 1 + \omega. \tag{5.13a, b}$$

Two different cases are possible:

(i) *No yielding occurs at the middle section* (Fig. 1b with $q_0 = 0$ and Fig. 1c). Solving eqn (5.8a) considered as an equality, as well as eqn (5.12) with q given by eqn (5.11a), we obtain

$$\xi = 2(s-1), \quad \omega = (s-1)/(2-s) \tag{5.14a, b}$$

$$q = -2a(s-1), \quad 4B_0/F_0L = 4a(s-1)^2 \tag{5.14c, d}$$

and the bounding quantity proves to be

$$\frac{1}{2}\frac{G}{\omega}\frac{4B_0}{F_0L} = 2a(s-1). \tag{5.15}$$

This is valid for $1 \le s \le s_1$, s_1 being the load for which yielding first occurs in the middle section, i.e.

$$s_1 = (\beta + 2)/3. \tag{5.16}$$

(ii) *Yielding occurs at the middle section* (Fig. 1b with $q_0 \neq 0$ and Fig. 1d). Solving eqns (5.8a, b), both considered as equalities, as well as eqn (5.12) with q given by eqn (5.11b), we deduce

$$\xi = \beta - s, \quad \omega = 2(s_c - s)/(2s - \beta), \tag{5.17a, b}$$

$$q = \frac{1}{4}a(\beta - s)^2/(s_c - s), \quad 4B_0/F_0L = a(\beta - s)^2, \tag{5.17c, d}$$

while the bounding quantity is now:

$$\frac{1}{2}\frac{G}{\omega}\frac{4B_0}{F_0L} = \frac{1}{4}a(\beta - s)^2/(s_c - s). \tag{5.18}$$

In Fig. 2 the real plastic work (5.4) and the bounding quantity (5.15) or (5.18) are plotted as functions of s for different values of $\beta (\beta = 1.5; 2.0 \text{ and } 2.5)$. It is shown that within the interval $(1, s_1)$ the bounding quantity is twice the real plastic work, while within the interval (s_1, s_c) the difference still increases with s and diverges for $s \to s_c$.

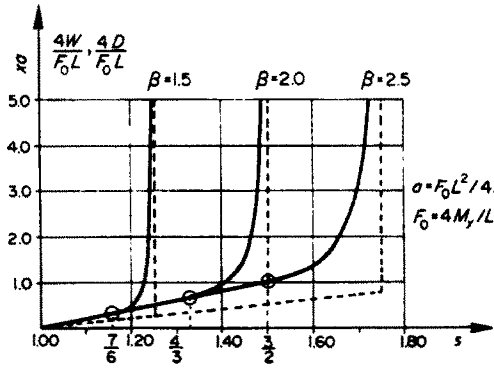

Fig. 2. Clamped beam loaded on the middle section: Comparison between the real plastic work W (dashed lines) and the upper bound D (solid lines) for different values of $\beta = M_y^-/M_y^+$.

6. CONCLUSION

Considering an elastic–perfectly plastic solid subjected to variable external actions which belong to a given domain but whose time history is unknown, we have formulated a bound on the dissipation energy which is produced locally at a given point of the body during the shakedown process. The method used is a perturbation method we have already applied in previous works[9–11]. Such method enables us to find optimal bounds by solving an analysis problem of finite plasticity or, equivalently, a convex minimization problem. This analysis problem may be interpreted as the holonomic version of the original shakedown problem suitably perturbed, and the above minimization problem thus constitutes an associated minimum principle. The holonomic solution causes the bounds to be expressed in a local form instead of an integral form.

The question to be answered is how good bounds the present method furnishes. The numerical example of Section 5, presented only for the sake of an illustration, does not give such an answer and further numerical applications are needed. Extensions and developments of the present method (for instance considering workhardening behaviour and/or dynamic loads) will be considered in future research work.

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APPENDIX

In this appendix the optimality conditions (4.6a, b)–(4.11) are deduced. First of all let the inequalities (4.4d, e) be transformed into equalities introducing appropriate slack variables, say z_α in $V+\Pi$ and z in V, i.e.

$$\varphi_\alpha(s_{ij}) + z_\alpha^2 = 0, \quad \forall \alpha \in (1, 2, \ldots, m), \quad \text{in } V, \tag{A.1}$$

$$g - G + z^2 = 0, \quad \text{in } V. \tag{A.2}$$

Denoting by w', v_i' and l_α', ($\alpha = 1, 2, \ldots, m$), Lagrangian variables which are defined in appropriate domains, we consider the Lagrangian functional

$$\Phi^* = \frac{G}{\omega}\int_V \frac{1}{2} A_{ijhk}\rho_{ij}\rho_{hk}\, \mathrm{d}V + \int_V \rho_{ij,j} v_i'\, \mathrm{d}V - \int_{S_1} \rho_{ij} n_j v_i'\, \mathrm{d}S$$
$$+ \int_\Pi \int_V [\varphi_\alpha(s_{ij}) + z_\alpha^2] l_\alpha'\, \mathrm{d}V\, \mathrm{d}\Pi + \int_V (g - G + z^2) w'\, \mathrm{d}V, \tag{A.3}$$

where g is given by eqns (4.1) and (4.2a) and φ_α, s_{ij} are given by eqns (4.5a–c). The first variation of Φ^*, after some rearrangements and using the equality

$$\int_V \delta\rho_{ij,j} v_i'\, \mathrm{d}V = \int_S v_i'\delta\rho_{ij} n_j\, \mathrm{d}S - \int_V \frac{1}{2}(v_{i,j}' + v_{j,i}')\delta\rho_{ij}\, \mathrm{d}V, \tag{A.4}$$

reads as in the following:

$$\delta\Phi^* = \left[-\frac{G}{\omega^2}\int_V \frac{1}{2} A_{ijhk}\rho_{ij}\rho_{hk}\, \mathrm{d}V + \int_V \int_\Pi \bar{\gamma} s_{ij} \frac{\partial f_\alpha}{\partial Q_{ij}} l_\alpha'\, \mathrm{d}\Pi\, \mathrm{d}V\right]\delta\omega$$
$$+ \int_V \rho_{ij,j}\delta v_i'\, \mathrm{d}V + \int_V \left[\frac{G}{\omega} A_{ijhk}\rho_{hk} + \int_\Pi g\frac{\partial f_\alpha}{\partial Q_{ij}} l_\alpha'\, \mathrm{d}\Pi - v_{i,j}'\right]\delta\rho_{ij}\, \mathrm{d}V$$
$$- \int_{S_1} \rho_{ij} n_j \delta v_i'\, \mathrm{d}S + \int_{S_2} \delta\rho_{ij} n_j v_i'\, \mathrm{d}S + \int_\Pi \int_V \left[f_\alpha(Q_{ij}) + z_\alpha^2\right]\delta l_\alpha'\, \mathrm{d}V\, \mathrm{d}\Pi$$
$$+ \left[\frac{1}{\omega}\int_V \frac{1}{2} A_{ijhk}\rho_{ij}\rho_{hk}\, \mathrm{d}V - \int_V w'\, \mathrm{d}V\right]\delta G + \int_V (\omega\bar{\gamma} - G + 1 + z^2)\delta w'\, \mathrm{d}V$$
$$+ 2\int_\Pi \int_V z_\alpha l_\alpha' \delta z_\alpha\, \mathrm{d}V\, \mathrm{d}\Pi + 2\int_V z w' \delta z\, \mathrm{d}V, \tag{A.5}$$

where for the sake of convenience we have set

$$Q_{ij} = g s_{ij}, \qquad \text{in } V + \Pi. \tag{A.6}$$

In order that $\delta\Phi^*$ be zero for any consistent variation of all the variables, the following field equations are to be satisfied:

$$\rho_{ij,j} = 0, \quad \text{in } V; \quad \rho_{ij} n_j = 0, \quad \text{on } S_1; \tag{A.7a, b}$$

$$\eta_{ij}' = \frac{1}{2}(v_{i,j}' + v_{j,i}'), \quad \text{in } V; \qquad v_i' = 0, \quad \text{on } S_2; \tag{A.8a, b}$$

$$\eta_{ij}' = \frac{G}{\omega} A_{ijhk}\rho_{hk} + \int_\Pi g\frac{\partial f_\alpha}{\partial Q_{ij}} l_\alpha'\, \mathrm{d}\Pi, \quad \text{in } V, \tag{A.8c}$$

$$f_\alpha(Q_{ij}) + z_\alpha^2 = 0, \quad z_\alpha l_\alpha' = 0, \quad \text{in } V + \Pi; \tag{A.9a, b}$$

$$\int_V \bar{\gamma} w'\, \mathrm{d}V + \int_V \bar{\gamma}\int_\Pi s_{ij}\frac{\partial f_\alpha}{\partial Q_{ij}} l_\alpha'\, \mathrm{d}\Pi\, \mathrm{d}V = \frac{G}{\omega^2}\int_V \frac{1}{2} A_{ijhk}\rho_{ij}\rho_{hk}\, \mathrm{d}V; \tag{A.10}$$

$$\omega\bar{\gamma} - G + 1 + z^2 = 0, \quad z w' = 0, \quad \text{in } V; \tag{A11}$$

$$\omega\int_V w'\, \mathrm{d}V = \int_V \frac{1}{2} A_{ijhk}\rho_{ij}\rho_{hk}\, \mathrm{d}V. \tag{A.12}$$

Let us now make the transformation

$$v_i = \frac{\omega}{G} v_i', \quad l_\alpha = \frac{\omega}{G} l_\alpha', \quad w = \frac{\omega}{G} w', \quad \eta_{ij} = \frac{\omega}{G}\eta_{ij}'. \tag{A.13a–d}$$

Then, in view of the equality

$$\frac{\partial\varphi_\alpha}{\partial s_{ij}} = g\frac{\partial f_\alpha}{\partial Q_{ij}}, \quad \forall\alpha \in (1, 2, \ldots, m), \quad \text{in } V, \tag{A.14}$$

and by elimination of the slack variables, eqns (A.7a, b) to (A.12) become respectively

$$\rho_{ij,j} = 0, \quad \text{in } V; \quad \rho_{ij} n_j = 0, \quad \text{on } S_1; \tag{A.15a, b}$$

$$\eta_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad \text{in } V; \quad v_i = 0, \quad \text{on } S_2; \tag{A.16a, b}$$

$$\eta_{ij} = A_{ijhk}\rho_{hk} + q_{ij}, \quad \text{in } V; \tag{A.16c}$$

$$\varphi_\alpha(s_{ij}) \le 0, \quad l_\alpha \ge 0, \quad \varphi_\alpha l_\alpha = 0, \quad \text{in } V + \Pi; \tag{A.17A–c}$$

$$\int_V \frac{1}{2} A_{ijhk}\rho_{ij}\rho_{hk}\,\mathrm{d}V = \omega \int_V \bar{\gamma} g^{-1} \int_\Pi s_{ij} d_{ij}\,\mathrm{d}\Pi\,\mathrm{d}V + \omega \int_V \bar{\gamma} w\,\mathrm{d}V \tag{A.18}$$

$$g - G \leq 0, \quad w \geq 0, \quad (g - G)w = 0, \quad \text{in } V, \tag{A.19a–c}$$

$$\int_V \frac{1}{2} A_{ijhk}\rho_{ij}\rho_{hk}\,\mathrm{d}V = G \int_V w\,\mathrm{d}V, \tag{A.20}$$

where we have set

$$d_{ij} = \frac{\partial \varphi_\alpha}{\partial s_{ij}} l_\alpha, \quad \text{in } V + \Pi, \tag{A.21a}$$

$$q_{ij} = \int_\Pi d_{ij}\,\mathrm{d}\Pi, \quad \text{in } V. \tag{A.21b}$$

The integration of the equality (A.19c) over the volume V gives

$$\int_V g w\,\mathrm{d}V = G \int_V w\,\mathrm{d}V. \tag{A.22}$$

Then, combining the latter with eqns (A.18) and (A.20) yields

$$\int_V \frac{1}{2} A_{ijhk}\rho_{ij}\rho_{hk}\,\mathrm{d}V = G\omega \int_V \bar{\gamma} g^{-1} \int_\Pi s_{ij} d_{ij}\,\mathrm{d}\Pi\,\mathrm{d}V. \tag{A.23}$$

Finally we observe that:

(a) If $w = w(x)$ is everywhere zero, the integral on the left side of eqn (A.20) vanishes, which implies that $\rho_{ij} = 0$ everywhere in V. This situation occurs when the elastic solution is everywhere inside the perturbed yield surface, i.e. $\varphi_\alpha(\sigma_{ij}^E) < 0$ (and hence $f_\alpha(\sigma_{ij}^E) < 0$). In other words, the body remains always elastic.

(b) If $w \neq 0$ at a point, there it follows that $G = g$ in accordance with eqn (A.19c). Therefore, the constant G proves to be

$$G = \max_{x \in V} g(x), \tag{A.24}$$

and w may be different from zero only where g takes its maximum value.

(c) The Langrangian variable w can be eliminated from the set of optimality conditions by simply substituting eqn (A.18) with eqn (A.23) and eqns (A.19a–c) and (A.20) with eqn (A.24). So we obtain the optimality conditions of Section 4.